Transient gravity wave response to an oscillating pressure

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The gravity-wave response of a semi-infinite liquid to the oscillating pressure $P\delta(x)\exp(i\omega t)$ is given in an asymptotic form that is uniformly valid through the transition zone that separates the dispersion-controlled precursor and the monochromatic steady state. The same problem has been considered previously by Stoker (1957), but his initial conditions were spurious, and he did not seek a uniformly valid asymptotic representation.

1. Statement of the problem

We consider the transient development of two-dimensional gravity waves on a semi-infinite body of liquid following the application of the pressure

$$p(x,t) = P\delta(x)\,e^{i\omega t}H(t), \tag{1.1}$$

or, more generally,

$$p(x,t) = \delta(x)\,F(t) \tag{1.2}$$

to the free surface $y = 0$. P denotes the complex amplitude of the total force per unit width (the imaginary parts of complex expressions are to be discarded in the final reckoning, according to the usual convention), $\delta(x)$ the Dirac delta function, ω the angular frequency, and $H(t)$ the Heaviside step function. Letting $\phi(x,y,t)$ denote the velocity potential, $\eta(x,t)$ the free-surface displacement, g the acceleration of gravity, and ρ the density of the liquid, we then have the following initial-value problem for the determination of ϕ and η:

$$\phi_{xx}+\phi_{yy} = 0, \tag{1.3}$$

with

$$\phi_y = \eta_t \quad \text{at} \quad y = 0, \tag{1.4}$$

$$\phi_t+g\eta = -p/\rho \quad \text{at} \quad y = 0, \tag{1.5}$$

and

$$\phi = \eta = 0 \quad \text{at} \quad t = 0. \tag{1.6}$$

The problem posed by (1.2)–(1.5), together with the initial conditions

$$\phi = \phi_t = 0 \quad \text{at} \quad t = 0, \tag{1.6S}$$

has been considered previously by Stoker (1957) with the implicit assertion that these initial conditions are identical with those of (1.6). In fact, $\eta(x, 0+) = 0$ implies $\phi_t(x,y,0+) = 0$ only if $p(x,0+) = 0$. The primary motivation of Stoker's analysis was to demonstrate that the limiting form of $\phi(x,y,t)$ as $t\to\infty$ is identical with Lamb's steady-state solution (1904) and that this limiting form satisfies

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a radiation condition as $|x| \to \infty$. As might have been expected, the substitution of the initial conditions (1.6S) for those of (1.6) has no effect on this conclusion.†

Our purpose in the present analysis, aside from correcting Stoker's formulation, is to examine in more detail the nature of the transient wave front. Remarking that the appropriate scales for t and x are $1/\omega$ and g/ω^2, we may anticipate the dominant features of this wave front as follows.

(*a*) If $g/\omega^2 \ll x \ll gt/\omega$ the free-surface displacement will approximate the asymptotic form of Lamb's (1904) steady-state solution, namely

$$\eta(x,t) \sim i(P\omega^2/\rho g^2)\exp\{i\omega[t-(\omega/g)\,x]\}, \tag{1.7}$$

corresponding to a monochromatic gravity wave advancing with the phase velocity g/ω.

(*b*) If $g/\omega^2 \ll gt/\omega \ll x$ the free-surface displacement will be essentially dispersive in character and will be given by the classical Cauchy–Poisson solution (Lamb 1932) for an impulse (in time) of complex amplitude

$$\int_0^\infty P\exp(i\omega t)\,dt = i(P/\omega),$$

namely
$$\eta(x,t) \sim i(Pg^{\frac{1}{2}}t^2/4\pi^{\frac{1}{2}}\rho\omega x^{\frac{5}{2}})\cos[(gt^2/4x)+\tfrac{1}{4}\pi]. \tag{1.8}$$

(*c*) The asymptotic solutions of (1.7) and (1.8) will be separated by a transition zone that advances with the group velocity $g/2\omega$ and has a width of $O(g\omega^{-\frac{3}{2}}t^{\frac{1}{2}})$ as $\omega^2x/g \to \infty$; see (4.9) below. The asymptotic form of the free-surface displacement in this zone will be

$$\eta(x,t) = (A+iB)\,(1.7)\,[1+O(\omega^2x/g)^{-\frac{1}{2}}], \tag{1.9}$$

where $A+iB$ may be regarded as the normalized complex amplitude of the motion in the transition zone. The envelope of the oscillatory motion in this zone then will be proportional to $(A^2+B^2)^{\frac{1}{2}}$.

We shall proceed by first determining (in § 2) a formal solution to (1.1)–(1.6). We then shall determine (in § 3) an asymptotic representation of η that is uniformly valid throughout the transition zone and that has the limiting forms of (1.7)–(1.9).

2. Formal solution

We choose as our starting point the Cauchy–Poisson solution, say $\phi = \Phi$, obtained by setting $F(t) = \delta(t)$ in (1.2)–(1.6), viz. (Lamb 1932)

$$\Phi(x,y,t) = -\frac{1}{\pi\rho}\int_0^\infty e^{ky}\cos(kx)\cos(\sigma t)\,dk, \tag{2.1}$$

where
$$\sigma = (gk)^{\frac{1}{2}}. \tag{2.2}$$

We then may construct the more general solution

$$\phi(x,y,t) = \int_0^t F(u)\,\Phi(x,y,t-u)\,du \tag{2.3}$$

by superposition.

† Stoker (1957) also posed the spurious initial conditions (1.6S) in his analysis of unsteady waves created by a prescribed pressure on the surface of a running stream. Wurtele (1955) has given a correct solution for a special case of the running-stream problem with results that have some similarity to those presented here.

Substituting the explicit time-dependence $F(t) = P\exp(i\omega t)$ into (2.3), we may place the result in the form

$$\phi = \frac{P}{\pi\rho}\left(\frac{\partial}{\partial t}+i\omega\right)\int_0^\infty e^{ky}\cos(kx)\left[\frac{\cos(\sigma t)-\cos(\omega t)}{\sigma^2-\omega^2}\right]dk. \tag{2.4}$$

Replacing the initial conditions (1.6) by (1.6S), we obtain

$$\phi = \frac{iP}{\pi\rho}\left(\frac{\partial}{\partial t}+i\omega\right)\int_0^\infty e^{ky}\cos(kx)\left[\frac{\omega\sin(\sigma t)-\sigma\sin(\omega t)}{\sigma(\sigma^2-\omega^2)}\right]dk$$
$$= i\omega\int_0^t (2.4)\,dt \tag{2.4S}$$

in place of (2.4); this is equivalent to Stoker's result (1957). Assuming the asymptotic time-dependence $\exp(i\omega t)$, the operators $i\omega$ and $\int_0^t (\;)\,dt$ cancel, and we have the anticipated result that (2.4) and (2.4S) are identical in the limit $t \to \infty$. We also observe that the integrands of both (2.4) and (2.4S) are bounded at $\sigma = \omega$.

We may construct the surface-displacement similarly,

$$\eta(x,t) = \int_0^t F(u)\,N(x,t-u)\,du, \tag{2.5}$$

where†

$$N(x,t) = -g^{-1}\lim_{y\to 0-}\Phi_t(x,y,t). \tag{2.6}$$

Substituting (2.1) into (2.6), integrating the result with respect to x in order to permit the passage to the limit $y = 0-$ prior to the integration, and introducing the change of variable $k = \sigma^2/g$, we obtain‡

$$N(x,t) = -\frac{2}{\pi\rho g}\frac{\partial}{\partial x}\int_0^\infty \sin\left(\frac{\sigma^2 x}{g}\right)\sin(\sigma t)\,d\sigma. \tag{2.7}$$

3. Asymptotic representation of surface displacement

We may obtain an asymptotic representation of $N(x,t)$ as $x\to\infty$ and $t = O(x)$ from the stationary-phase approximation§ (already cited in (1.8) above)

$$N(x,t) \sim \frac{g^{\frac{1}{2}}t^2}{4\pi^{\frac{1}{2}}\rho x^{\frac{5}{2}}}\cos\left(\frac{gt^2}{4x}+\tfrac{1}{4}\pi\right)+O(x^{-\frac{3}{2}}). \tag{3.1}$$

We have assumed $x > 0$, but we also could replace x by $|x|$ in (3.1) et seq. Substituting (3.1) and $F(t) = P\exp(i\omega t)$ into (2.5), we obtain

$$\eta(x,t) \sim \frac{Pg^{\frac{1}{2}}}{4\pi^{\frac{1}{2}}\rho x^{\frac{5}{2}}}\int_0^t e^{i\omega(t-u)}\cos\left(\frac{gu^2}{4x}+\tfrac{1}{4}\pi\right)u^2\,du. \tag{3.2}$$

† We may also deduce (2·5) and (2·6) from (2.1) and (2.2) through (1.5) after integration by parts.

‡ Cf. Lamb's (1932) result §239(31).

§ Lamb (1932, §239(38)). Lamb does not state the error term, but it follows directly from his analysis.

Introducing the dimensionless variables

$$\xi = \omega^2 x/g \quad \text{and} \quad \tau = \omega t, \tag{3.3}$$

the wave-front parameter

$$\theta = \tau/2\xi = gt/2\omega x, \tag{3.4}$$

and the change of variable $u = (2\xi/\omega)\varphi$, we may rewrite (3.2) in the form

$$\eta(x,t) = (P\omega^2/\rho g^2)\,\psi(\xi,\tau), \tag{3.5}$$

where

$$\begin{aligned}\psi(\xi,\tau) &= 2(\xi/\pi)^{\frac{1}{2}} \int_0^\theta e^{2i\xi(\theta-\varphi)} \cos(\xi\varphi^2 + \tfrac{1}{4}\pi)\,\varphi^2 d\varphi + O(\xi^{-\frac{3}{2}}) \\ &= (\xi/\pi)^{\frac{1}{2}} e^{2i\xi\theta}\left[e^{i(\frac{1}{4}\pi-\xi)}\int_0^\theta e^{i\xi(\varphi-1)^2}\varphi^2 d\varphi + e^{i(\xi-\frac{1}{4}\pi)}\int_0^\theta e^{-i\xi(\varphi+1)^2}\varphi^2 d\varphi\right] + O(\xi^{-\frac{3}{2}}).\end{aligned} \tag{3.6}$$

Considering first the last integral in (3.6), we may integrate by parts to obtain

$$\int_0^\theta e^{-i\xi(\varphi+1)^2}\varphi^2 d\varphi = [i\theta^2/2\xi(\theta+1)]\,e^{-i\xi(\theta+1)^2} + O(\xi^{-2}). \tag{3.7}$$

The remaining integral has a point of stationary phase at $\varphi = 1$ if $\theta > 1$. We therefore introduce the change of variable $v = \varphi - 1$ and proceed as follows:

$$\begin{aligned}\int_0^\theta e^{i\xi(\varphi-1)^2}\varphi^2 d\varphi &= (1+\tfrac{1}{2}v)\frac{e^{i\xi v^2}}{i\xi}\Bigg|_{-1}^{\theta-1} + \left(1-\frac{1}{2i\xi}\right)\int_{-1}^{\theta-1} e^{i\xi v^2}dv \\ &= \frac{1}{2}\left(\frac{\pi}{\xi}\right)^{\frac{1}{2}} e^{\frac{1}{4}i\pi} + \int_0^{\theta-1} e^{i\xi v^2}dv + \left(\frac{\theta+1}{2i\xi}\right)e^{i\xi(\theta-1)^2} + O(\xi^{-\frac{3}{2}}).\end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we obtain

$$\begin{aligned}\psi &= i\,e^{i\xi(2\theta-1)}\left[\tfrac{1}{2} + \pi^{-\frac{1}{2}} e^{-\frac{1}{4}i\pi}\int_0^{\xi^{\frac{1}{2}}(\theta-1)} e^{iw^2}dw\right] \\ &\quad + \tfrac{1}{2}i(\pi\xi)^{-\frac{1}{2}}[\theta^2(\theta+1)^{-1}e^{-i(\xi\theta^2+\frac{1}{4}\pi)} - (\theta+1)\,e^{i(\xi\theta^2+\frac{1}{4}\pi)}] + O(\xi^{-\frac{3}{2}}).\end{aligned} \tag{3.9}$$

This last result is uniformly valid with respect to θ in the neighbourhood of $\theta = 1$, i.e. in the neighbourhood of the interface $x = (g/2\omega)t$ that advances with the group velocity $g/2\omega$. If θ is bounded away from 1 we may introduce the asymptotic approximation

$$\int_0^{\xi^{\frac{1}{2}}(\theta-1)} e^{iw^2}dw = \tfrac{1}{2}\pi^{\frac{1}{2}} e^{\frac{1}{4}i\pi}\,\mathrm{sgn}\,(\theta-1) - \tfrac{1}{2}i(\theta-1)^{-1}\xi^{-\frac{1}{2}} e^{i\xi(\theta-1)^2} + O(\xi^{-\frac{3}{2}}) \tag{3.10}$$

to obtain

$$\psi = i\,e^{i\xi(2\theta-1)}H(\theta-1) + \tfrac{1}{2}\theta^2(\pi\xi)^{-\frac{1}{2}}[(\theta-1)^{-1}e^{i(\xi\theta^2-\frac{1}{4}\pi)} + (\theta+1)^{-1}e^{-i(\xi\theta^2-\frac{1}{4}\pi)}] + O(\xi^{-\frac{3}{2}}). \tag{3.11}$$

Returning now to the original variables through (3.3)–(3.5), we may transform (3.11) to

$$\begin{aligned}\eta &= i(P\omega^2/\rho g^2)\exp\{i\omega[t-(\omega/g)x]\}\,H[t-(2\omega/g)x] \\ &\quad + (Pg^{\frac{1}{2}}/4\pi^{\frac{1}{2}}\rho\omega)\,t^2x^{-\frac{5}{2}}[(gt/2\omega x)^2-1]^{-1} \\ &\quad \times\{(gt/2\omega x)\cos[(gt^2/4x)-\tfrac{1}{4}\pi] + i\sin[(gt^2/4x)-\tfrac{1}{4}\pi]\} \\ &\quad \times[1+O(\omega^2x/g)^{-\frac{1}{2}}].\end{aligned} \tag{3.12}$$

Considering the limits $t \to \infty$ for fixed x and $x \to \infty$ for fixed t, we then may confirm the results anticipated in (1.7) and (1.8).

4. Asymptotic envelope

Substituting (3.9) into (3.5) and returning to the original variables through (3.3), we may place the result in the form

$$\eta(x,t) = i(P\omega^2/\rho g^2)\,[A(u)+iB(u)]\exp\{i\omega[t-(\omega/g)\,x]\}\,[1+O(\omega^2 x/g)^{-\frac{1}{2}}], \quad (4.1)$$

$$A+iB = \tfrac{1}{2}[1+C+S+i(S-C)], \quad (4.2)$$

$$C+iS = \int_0^u e^{\frac{1}{2}i\pi v^2}\,dv, \quad (4.3)$$

$$u = (2\xi/\pi)^{\frac{1}{2}}\,(\theta-1) = (g/2\pi x)^{\frac{1}{2}}\,[t-(2\omega/g)\,x]. \quad (4.4)$$

We observe that the Fresnel integrals, C and S, are both odd functions of u.

The result (4.1) describes the asymptotic (as $\omega^2 x/g \to \infty$) form of the free surface displacement in terms of a travelling wave that has the frequency ω, the phase velocity g/ω, the slowly changing (relative to ω) envelope

$$(|P|\,\omega^2/\rho g^2)\,R(u),$$

where

$$R(u) = (A^2+B^2)^{\frac{1}{2}}, \quad (4.5)$$

and the slowly changing phase angle $\frac{1}{2}\pi+\tan^{-1}(B/A)$ relative to that of the complex amplitude P. The centre of the normalized envelope $R(u)$, defined by $u=0$, advances with the group velocity $g/2\omega$; its distribution is plotted in figure 1.

Regarded as a timewise envelope—i.e. as the envelope measured by an observer at a fixed point x—$R(u)$ rises monotonically to a maximum of 1·17 at $u = 1{\cdot}2$ and then enters an oscillatory epoch, in which the asymptotic behaviour is given by (cf. (3.10))

$$R(u) \sim 1+2^{-\frac{1}{2}}(\pi u)^{-1}\sin\left(\tfrac{1}{2}\pi u^2-\tfrac{1}{4}\pi\right)+O(u^{-2}). \quad (4.6)$$

We emphasize, however, that (4.6) is a consistent approximation only in so far as $\theta \ll 1$ and $\xi^{\frac{1}{2}}(\theta-1) \gg 1$; if $\xi^{\frac{1}{2}}(\theta-1) \gg 1$ but θ is not small the second term in (4.6) is of the same order as terms already neglected—cf. (3.11). We may define the rise-time T as the time for R to rise from 0·1 to its first maximum,

$$T \doteqdot 9(x/g)^{\frac{1}{2}}. \quad (4.7)$$

We remark that T is independent of the frequency ω and that, by hypothesis, $\omega T \gg 1$ (by virtue of which we have described the envelope as slowly changing).

We also may regard $R(-u)$ as the spacewise envelope at a fixed time, since

$$-u = 2\omega^{\frac{3}{2}}g^{-1}(\pi t)^{-\frac{1}{2}}\,[x-(g/2\omega)\,t]\,[1+O(\omega^2 x/g)^{-\frac{1}{2}}], \quad (4.8)$$

which is within the approximation already invoked in (4.1). Viewed in this manner, the wave-front envelope rises monotonically to its maximum value as x decreases from $+\infty$ and then tends in an oscillatory fashion to its steady-state value. We may define the width X of the transition zone, analogously with T, as the distance between the point at which the precursor reaches 0·1 of the steady-state amplitude and its maximum amplitude,

$$X \doteqdot 3g\omega^{-\frac{3}{2}}t^{\frac{1}{2}}. \tag{4.9}$$

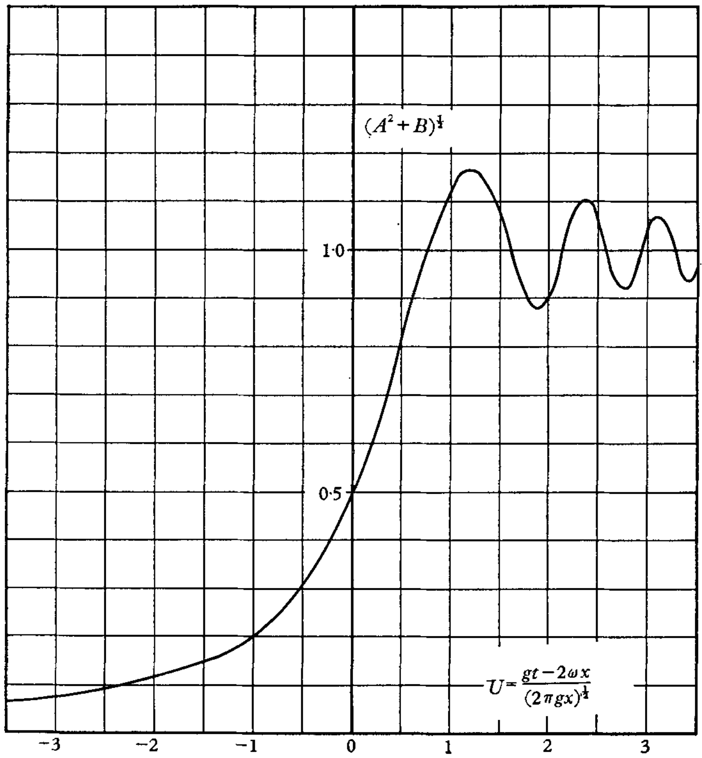

FIGURE 1. The envelope $R(u)$, as given by equations (4.2)–(4.5).

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